GALOIS LINES FOR NORMAL ELLIPTIC SPACE CURVES, II

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Abstract. For each linearly normal elliptic curve C in $\mathbb{P}^3$, we determine Galois lines and their arrangement. The results are as follows: the curve C has just six V_4-lines and in case $j(C)=1$, it has eight Z_4-lines in addition. The V_4-lines form the edges of a tetrahedron, in case $j(C)=1$, for each vertex of the tetrahedron, there exist just two Z_4-lines passing through it. We obtain as a corollary that each plane quartic curve of genus one does not have more than one Galois point.

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1. Introduction

This is a continuation of [1], where we found three V_4-lines for each linearly normal elliptic curve C in $\mathbb{P}^3$, and four Z_4-lines for such curve C with $j(C)=1$. However, those lines are not all the ones. In this article we determine all Galois lines and describe their arrangement. First let us recall the definition of Galois lines briefly.

Let k be the ground field of our discussion, we assume it to be algebraically closed, later we assume it the field $\mathbb{C}$ of complex numbers. Let C be a smooth irreducible non-degenerate curve of degree d in the projective three space $\mathbb{P}^3$ and ℓ a line in $\mathbb{P}^3$ not meeting C. Let $\pi_\ell : \mathbb{P}^3 \dashrightarrow l_0$ be the projection with center ℓ, where ℓ_0 is a line not meeting ℓ. Restricting π_ℓ to C, we get a surjective morphism $\pi_\ell|_C : C \longrightarrow l_0$ and hence an extension of fields $(\pi_\ell|_C)^* : k(\ell_0) \hookrightarrow k(C)$, where $[k(C):k(\ell_0)]=d$. Note that the extension of fields does not depend on ℓ_0, but on ℓ.

Definition 1. The line ℓ is said to be a Galois line for C if the extension $k(C)/k(\ell_0)$ is Galois, or equivalently, if $\pi_\ell|_C$ is a Galois covering. In this case $\mathrm{Gal}(k(C)/k(\ell_0))$ is said to be the Galois group for ℓ and denoted by G_ℓ.

If ℓ is the Galois line, then each element $\sigma \in G_l$ induces an automorphism of C over ℓ_0. We denote it by the same letter σ. Hereafter, assume C is linearly normal, i.e., the hyperplanes cut out the complete linear series $|\mathcal{O}_C(1)|$. Then, the automorphism σ can be extended to a projective transformation of $\mathbb{P}^3$, which will be also denoted by the same letter σ.

We use the following notation and convention:

· V_4 : the Klein 4-group
· Z_4 : the cyclic group of order four
· $\sim$: the linear equivalence of divisors
· $\mathrm{Aut}(C)$: the automorphism group of C

· $\mathcal{L}(D) := \{ f \in k(C) \setminus \{0\} \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$, where $\mathrm{div}(f)$ is the divisor of f and D is a divisor on C.
· $\langle \cdots \rangle$: the group generated by the set $\{\cdots\}$ or the linear subvariety spanned by the set $\{\cdots\}$
· $V(F)$: the variety defined by $F = 0$
· $C \cdot H$: the intersection divisor of C and H on C, where H is a plane.
· ℓ_{PQ} : the line passing through P and Q

2. Statement of Results

We assume $k = \mathbb{C}$ and use the same notation as in [1].

Definition 2. When ℓ is a Galois line for C and $G_\ell \cong V_4$ (resp. Z_4), we call ℓ a V_4 (resp. Z_4)-line.

There exist V_4-lines for the curve which is given by an intersection of hypersurfaces as follows.

Lemma 1. *Suppose S_1 and S_2 are irreducible quadratic surfaces in $\mathbb{P}^3$ satisfying the following conditions*:

(1) *S_i $(i = 1, 2)$ has a singular point Q_i and $Q_1 \neq Q_2$.*
(2) *$S_1 \cap S_2$ is a smooth curve Δ.*
(3) *The line ℓ passing through Q_1 and Q_2 does not meet Δ.*

Then, Δ is a linearly normal elliptic curve and ℓ is a V_4-line for Δ.

Let C be a linearly normal elliptic curve in $\mathbb{P}^3$. Then, there exists a divisor D of degree four on an elliptic curve E such that C is given by an embedding of E associated with the complete linear system $|D|$. Note that C can be expressed as an intersection of two quadratic surfaces.

Lemma 2. *There exist just four irreducible quadratic surfaces S_i ($0 \leq i \leq 3$) such that each S_i has a singular point and contains C. Let Q_i be the unique singular point of S_i. Then the four points are not coplanar.*

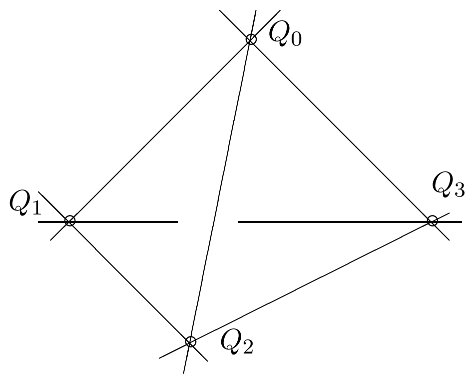
Remark 3. Let $\pi_Q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection with center $Q \in \mathbb{P}^3 \setminus C$. If π_Q induces a 2 to 1 morphism from C onto its image in Lemma 2, then Q coincides with one of Q_i.

The main theorem is stated as follows:

Theorem 1. *For each linearly normal elliptic curve in $\mathbb{P}^3$, there exist four non-coplanar points Q_i $(0 \leq i \leq 3)$ such that the lines passing through each two of them are V_4-lines for C. Namely, all the V_4-lines form the six edges of a tetrahedron. Further, if the Weierstrass normal form of E is given by $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$, then we can present explicitly the coordinates of Q_i (by taking a suitable coordinates of $\mathbb{P}^3$) as follows:*

$$Q_0 = (0 : 0 : 0 : 1) \text{ and } Q_i = (1 : -c_i : e_i : 0), \ (i = 1, 2, 3),$$

where $c_i = {e_i}^2 + e_j e_k$ such that $\{i, j, k\} = \{1, 2, 3\}$.



Remark 4. In the case of an elliptic curve E in $\mathbb{P}^2$ it has a Galois point if and only if $j(E) = 0$, and then it has just three Z_3-points.

In the case where the j-invariant $j(C) = 1$, there exists an automorphism of order four with a fixed point. This curve has the other Galois lines as follows.

Theorem 2. *Under the same assumption as in Theorem 1, if $j(C) = 1$, then there exist eight Z_4-lines (in addition to the V_4-lines). To state in more detail, for each vertex Q_i $(0 \leq i \leq 3)$ of the tetrahedron in Theorem 1, there exist two Z_4-lines passing through it. Therefore, for each vertex, there exist three V_4-lines and two Z_4-lines passing through it and the total number of Galois lines is fourteen. Two Z_4-lines do not meet except at one of the vertices.*

Let Σ be the set of six V_4-lines in Theorem 1. In the case where $j(C) = 1$ let Σ' be the set of eight Z_4-lines in Theorem 2. The following corollary is an answer to the question for the case of outer Galois point [3, Theorem 2].

Corollary 5. *For a plane quartic curve Γ with genus one, the number of (outer) Galois points is at most one. If Γ has the Galois point, then the Galois group G is isomorphic to V_4 or Z_4. Further, if $G \cong V_4$ (resp. Z_4), then Γ is obtained by a projection $\pi_Q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center Q, where $Q \in \Sigma$ (resp. $Q \in \Sigma'$) such that $Q \neq Q_i$ $(0 \leq i \leq 3)$.*

Remark 6. Different from the case of the space quartic curve, a plane quartic curve of genus one does not necessarily have a Galois point.

Remark 7. Since C is given by the embedding associated with a complete linear system and has a Galois line, the embedding is called a Galois embedding, which has been defined in [6].

3. Proofs

First we prove Lemma 1. It is easy to see that Δ has genus one and $\dim \mathrm{H}^0(\Delta,\ \mathcal{O}_\Delta(1)) = 4$. Hence Δ is a linearly normal elliptic curve. Let π_{Q_i} be the projection $\mathbb{P}^3 \dashrightarrow P^2$ with center Q_i $(i = 1,\ 2)$ and put $\Delta_i = \pi_{Q_i}(\Delta) \subset \mathbb{P}^2$ and $R_i = \pi_{Q_i}(\ell \setminus \{Q_i\})$. Then Δ_i is a conic and R_i is a point not on Δ_i. Let ϖ_{R_i} be the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with center R_i. Restricting ϖ_{R_i} to Δ_i, we get a surjective morphism $\varpi_{R_i}|_{\Delta_i} : C_i \longrightarrow \mathbb{P}^1$. Therefore we have two morphisms

$$\pi_i = \varpi_{Q_i} \cdot \pi_{Q_i} : \Delta \longrightarrow \mathbb{P}^1$$

of degree four. They coincide with the restriction of the projection $\pi_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. Note that $k(\Delta_1)$ and $k(\Delta_2)$ are distinct subfields of $k(\Delta)$ and $[k(\Delta) : k(\Delta_i)] = [k(\Delta_i) : k(\mathbb{P}^1)] = 2$. We infer that $k(\Delta)$ is a V_4-extension of $k(\mathbb{P}^1)$, hence $\pi_\ell|_\Delta$ is a V_4-Galois covering. This proves Lemma 1.

Fix a universal covering $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{L}$, where $\mathcal{L}$ is the lattice in $\mathbb{C}$ defining a complex torus. We assume $\mathcal{L} = \mathbb{Z} + \mathbb{Z}\omega$, where $\Im\omega > 0$. Let $\wp(z)$ be the Weierstrass $\wp$-function with respect to $\mathcal{L}$. Then, the map $\varphi : \mathbb{C} \longrightarrow E$ defined by $\varphi(z) = (\wp(z) : \wp'(z) : 1)$, induces an isomorphism $\bar{\varphi} : \mathbb{C}/\mathcal{L} \longrightarrow E$. The defining equation of the elliptic curve E is the Weierstrass normal form $y^2 = 4x^3 + px + q$. We assume it to be factored as $4(x - e_1)(x - e_2)(x - e_3)$. Put $P_\alpha = \varphi(\alpha)$ for $\alpha \in \mathbb{C}$. Denote by $+$ the sum of divisors on E and, at the same time, the sum of complex numbers. For example, $P_\alpha + P_\beta$ and $\alpha + \beta$ denote the sum of divisors and complex numbers respectively.

Lemma 8. *We have the linear equivalence of divisors on E:*

$$P_\alpha + P_\beta \sim P_{\alpha+\beta} + P_0.$$

Proof. This may be well-known. See, for example, [2, Ch. IV, Theorem 4,13B]. □

Lemma 9. *Let D be the divisor of degree four on E. By taking a suitable translation τ on E, we have $\tau^*(D) \sim 4P_0$.*

Proof. Suppose $D = \sum_{i=1}^4 P_{\alpha_i}$. Then, take $\beta = -\sum_{i=1}^4 \alpha_i/4$. Let τ be the translation on E induced from the one $z \mapsto z + \beta$ on $\mathbb{C}$. Then we have $\tau^*(D) = \sum_{i=1}^4 P_{\alpha_i+\beta}$. Using Lemma 8, we get $\tau^*(D) \sim 4P_0$. □

Let D be a hypeplane section of C. Applying Lemma 9, we see that there exists an elliptic curve C_0 in $\mathbb{P}^3$ given by the embedding associated with $|4P_0|$ and an isomorphism $\psi : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ satisfying that $\psi(C_0) = C$ and $4P_0 \sim \psi^*(D)$. So that we have the following lemma.

Lemma 10. *We can assume C is given by the embedding associated with $|4P_0|$.*

Therefore it is sufficient for our purpose to consider the curve embedded by $|4P_0|$. Let $\phi : E \longrightarrow C \subset \mathbb{P}^3$ be the embedding of E associated with $|4P_0|$.

$$\begin{tikzcd}
\mathbb{C} \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & & \\
\mathbb{C}/\mathcal{L} \arrow[r, "\bar{\varphi}"'] & E \arrow[r, "\phi"'] & C \subset \mathbb{P}^3
\end{tikzcd}$$

In order to study the number and arrangement of Galois lines, we provide some lemmas. Let $\mathcal{S}$ and $\mathcal{G}$ be the set of Galois lines for C and the set of subgroups of

Aut(C) respectively. Since a Galois line ℓ determine the Galois group G_ℓ in Aut(C) uniquely, we can define the following map.

Definition 3. We define an arrangement-map $\rho : \mathcal{S} \longrightarrow \mathcal{G}$ by $\rho(\ell) = G_\ell$.

We study the map ρ in detail. Note that each element of G_ℓ can be extended to a projective transformation. That is, we have a faithful representation $r : G_\ell \longrightarrow PGL(3, \mathbb{C})$.

Lemma 11. *The map ρ is injective.*

Proof. For two elements ℓ_i of $\mathcal{S}$ $(i = 1, 2)$, suppose $\rho(\ell_1) = \rho(\ell_2)$ and $\ell_1 \neq \ell_2$. Then, the following two cases take place:

(i) $\ell_1 \cap \ell_2$ consists of one point P.
(ii) $\ell_1 \cap \ell_2 = \emptyset$.

In the case (i), for a general point $Q \in C$, put $H_{iQ} = \langle \ell_i, Q\rangle$ $(i = 1, 2)$: the plane spanned by ℓ_i and Q. Since $G_{\ell_1} = G_{\ell_2}$, we have $H_{1Q} \cap \ell_0 = H_{2Q} \cap \ell_0 = \{R\}$, where ℓ_0 is the line defined in Introduction. Further, since $\pi_{\ell_1}(H_{1Q} \cap C) = \pi_{\ell_2}(H_{2Q} \cap C) = R$, the set of four points $H_{1Q} \cap C$ is equal to that of $H_{2Q} \cap C$ and they lie on the line $H_{1Q} \cap H_{2Q}$, which passes through P. This implies C is contained in the plane spanned by ℓ_0 and P. Since C is assumed to be non-degenerate, this is a contradiction. Next we treat the case (ii). Similarly, for a general point $Q \in C$, put $H_{iQ} = \langle \ell_i,\ Q\rangle$. Then, by the same argument as above, the four points $H_{1Q} \cap C$ and $H_{2Q} \cap C$ lie on the line $H_{1Q} \cap H_{2Q}$. Thus C is contained in a rational normal scroll Σ. However, $H_{iQ} \cap \Sigma$ is a line, so that Σ must be a plane. This is a contradiction. □

We present a criterion when $G \subset$ Aut(C) can be the image of an element of $\mathcal{S}$. See [6, Theorem 2.2] for a similar one. Hereafter we use the notation $P_\alpha{}' = \phi(P_\alpha) = (\phi\varphi)(\alpha) \in C$ for brevity.

Lemma 12. *A subgroup $G = \{\sigma_1, \ldots, \sigma_4\}$ of Aut(C) is an image of ρ if and only if G satisfies the following condition ($\diamondsuit$):*

($\diamondsuit$) *For each point $Q \in C$ the divisor $\sum_{i=1}^4 \sigma_i(Q)$ is linearly equivalent to $4P_0{}'$ and C/G is a rational curve.*

Proof. If $G = \rho(\ell)$, then clearly $C/G \cong \mathbb{P}^1$. Take a plane H satisfying that $H \supset \ell$ and $H \ni Q$. By definition the point $\sigma_i(Q)$ $(1 \le i \le 4)$ lies on H, hence the divisor is linearly equivalent to $4P_0{}'$. Conversely, for a point $Q \in C$, put $D = \sum_{i=1}^4 \sigma_i(Q)$. By assumption we have $D \sim 4P_0{}'$, hence G acts on $\mathrm{H}^0(C,\ \mathcal{O}_C(1))$. Therefore each element of G can be extended to a projective transformation. Letting $\pi : C \longrightarrow C/G \cong \mathbb{P}^1$, we take independent sections s_0 and s_1 of $\mathrm{H}^0(\mathbb{P}^1,\ \mathcal{O}_{\mathbb{P}^1}(1))$ and put $\widetilde{s_i} = \pi^*(s_i)$ $(i = 1, 2)$. Then we have $\sigma^*(\widetilde{s_i}) = \widetilde{s_i}$. Taking a basis of $\mathrm{H}^0(C,\ \mathcal{O}_C(1))$ containing $\widetilde{s_1}$ and $\widetilde{s_2}$, we obtain a Galois line ℓ such that $\rho(\ell) = G$. □

We study whether $\ell_1 \cap \ell_2 = \emptyset$ or $\neq \emptyset$ by observing $G_{\ell_1} \cap G_{\ell_2}$ in Aut(C).

Lemma 13. *Suppose ℓ_1 and ℓ_2 are distinct Galois lines. Then, the following two cases take place.*

(1) *If $\ell_1 \cap \ell_2 = \emptyset$, then $G_{\ell_1} \cap G_{\ell_2} = \{\mathrm{id}\}$ in* Aut(C).
(2) *If $\ell_1 \cap \ell_2$ is a point P, then it is a singular point of some quadratic surface containing C, Further, we have $G_{\ell_1} \cap G_{\ell_2} = \langle\sigma\rangle$, where σ has order two and has a fixed point as an automorphism of C.*

Proof. Take an element $\sigma \in G_{\ell_1} \cap G_{\ell_2}$. It can be extended to a projective transformation. Since every plane H_i containing ℓ_i is invariant by σ, we infer $\sigma(\ell_i) = \ell_i$ (i=1, 2). Therefore, for each hyperplane $H_1 \supset \ell_1$, if $H_1 \cap \ell_2 = \{Q\}$, then $\sigma(Q) = Q$, i.e., $\sigma|_{\ell_2} = \mathrm{id}$. By the same argument we also have $\sigma|_{\ell_1} = \mathrm{id}$. Since $\ell_1 \cap \ell_2 = \emptyset$, σ is identity on $\mathbb{P}^3$. Next we treat the second case. Suppose $\ell_1 \cap \ell_2$ consists of one point P. Then, for each point $Q \in C$, put $H_{iQ} = \langle \ell_i, Q \rangle$ and $\ell_Q = H_{1Q} \cap H_{2Q}$. Since $H_{iQ} \supset \ell_Q$ for $i = 1$ and 2, we have $\sigma(Q) \in \ell_Q$. Therefore C is contained in the cone passing through P. Clearly the order of σ is two. Since the quotient curve $C/\langle \sigma \rangle$ is isomorphic to ℓ_0, the σ has a fixed point in C. □

From Lemma 13 we infer the following remark.

Remark 14. Let ℓ be a Galois line and take a point $P \in \ell$. Let $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be a projection with center P. If P is not the vertex of the tetrahedron, then $\pi_P(\ell \setminus \{P\})$ is a Galois point for the quartic curve $\pi_P(C)$. However, if P is the one, then $\pi_P|_C$ turns out to be a 2 to 1 morphism onto its image and $\pi_P(C)$ is a conic in $\mathbb{P}^2$.

Hereafter we denote by σ_i $(0 \le i \le 3)$ an automorphism of E such that the representation on $\mathbb{C}$ is

$$\sigma_0(z) = -z, \quad \sigma_1(z) = -z + \frac{1}{2}, \quad \sigma_2(z) = -z + \frac{\omega}{2}, \quad \sigma_3(z) = -z + \frac{1+\omega}{2}.$$

Lemma 15. *The number of V_4-lines is at most six.*

Proof. Suppose C has a V_4-line ℓ. Then, let H be a plane containing ℓ and $P_0{}'$. Since $\pi_\ell|_C : C \longrightarrow \mathbb{P}^1$ is a V_4-covering, the intersection divisor $H \cdot C$ on C can be expressed in one of the following two types:

(i) $H \cdot C = 2P_0{}' + 2P_\gamma{}'$

(ii) $H \cdot C = P_0{}' + P_{\gamma_1}{}' + P_{\gamma_2}{}' + P_{\gamma_3}{}'$.

Suppose $G = \langle \sigma, \tau \rangle$, where

$$\sigma(z) = -z + \alpha \text{ and } \tau(z) = z + \beta \tag{1}$$

on the universal covering $\mathbb{C}$, where $2\beta \equiv 0 \pmod{\mathcal{L}}$ and $\beta \not\equiv 0 \pmod{\mathcal{L}}$. The case (i) (resp. (ii)) occurs when $\alpha \equiv 0 \pmod{\mathcal{L}}$ (resp. $\alpha \not\equiv 0 \pmod{\mathcal{L}}$) in (1). We consider the possibility of $\alpha \not\equiv 0$, i.e., we treat the case (ii). Since $H \cdot C$ is invariant by the action of G, it can be expressed as $P_0{}' + P_\alpha{}' + P_\beta{}' + P_{\alpha+\beta}{}'$. Since this is linearly equivalent to $4P_0{}'$, we infer

$$P_\alpha + P_\beta + P_{\alpha+\beta} \sim 3P_0 \tag{2}$$

on E. The left hand side of (2) is linearly equivalent to $P_{2(\alpha+\beta)} + 2P_0$ by Lemma 8. Therefore we have $P_{2(\alpha+\beta)} \sim P_0$. This implies $2(\alpha+\beta) \equiv 0 \pmod{\mathcal{L}}$, i.e., $2\alpha \equiv 0 \pmod{\mathcal{L}}$. Then, let us find the distinct subgroups G of $\mathrm{Aut}(C)$ such that G is generated by order two elements. By taking two from $\sigma_i (0 \le i \le 3)$, we have six subgroups $G_{ij} = \langle \sigma_i, \ \sigma_j \rangle$, where $0 \le i < j \le 3$. Clearly $G_{ij} \cong V_4$. For example, $G_{12} = \{\mathrm{id}, \ \sigma_1, \ \sigma_2, \ \sigma_1\sigma_2\}$, where $(\sigma_1\sigma_2)(z) = z + (1+\omega)/2$. □

Lemma 16. *Putiing $a_i = (e_i - e_j)(e_i - e_k)$, we have*

$$\sigma_0{}^*(x) = x, \quad \sigma_0{}^*(y) = -y$$

and

$$\sigma_i{}^*(x) = \frac{a_i}{x - e_i} + e_i, \quad \sigma_i{}^*(y) = \frac{a_i}{(x-e_i)^2} y, \ \textit{where } 1 \le i \le 3.$$

Proof. Since $x = \wp(z)$ and $y = \wp'(z)$, we can prove them by using the the addition formulas of $\wp$ and $\wp'$:

$$\begin{aligned}
\wp(z_1 + z_2) &= -\wp(z_1) - \wp(z_2) + \tfrac{1}{4}\left(\tfrac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 \text{ and} \\
\wp'(z_1 + z_2) &= \tfrac{-1}{\wp(z_1) - \wp(z_2)} \left[\wp'(z_1) \left\{ (-\wp(z_1) - 2\wp(z_2)) + \tfrac{1}{4}\left(\tfrac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 \right\} \right. \\
&\quad \left. + \wp'(z_2) \left\{ (2\wp(z_1) + \wp(z_2) - \tfrac{1}{4}\left(\tfrac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 \right\} \right]
\end{aligned}$$

□

Since $\mathcal{L}(4P_0) = \langle 1, \ x^2, \ x, \ y \rangle$, we can assume the curve C is given by the embedding $\phi(x, y) = (1 : x^2 : x : y)$. Let $(X : Y : Z : W)$ be a set of homogeneous coordinates on $\mathbb{P}^3$. Then the ideal of C is generated by

$$F_1 = XY - Z^2 \text{ and } F_2 = 4YZ + pXZ + qX^2 - W^2.$$

Lemma 17. *Using the same notation $G_{ij} = \langle \sigma_i, \ \sigma_j \rangle$ as in the proof of Lemma 15, we denote by $K_{ij} = k(x, y)^{G_{ij}}$ the fixed subfield of $k(x, y)$ by G_{ij}. Then we have*

$$K_{0i} = k\left(\frac{x^2 + c_i}{x - c_i}\right), \ \textit{where } 1 \le i \le 3$$

and

$$K_{ij} = k\left(\frac{y}{c_k + 2e_k x - x^2}\right), \ \textit{where } 1 \le i < j \le 3 \textit{ and } (k-i)(k-j) \ne 0.$$

In particular, the Galois lines which correspond to G_{0i} and G_{ij} by the arrangement-map ρ are

$$Y + c_i X = Z - e_i X = 0 \ \textit{ and } \ c_k X - Y + 2e_k Z = W = 0$$

respectively.

Proof. By making use of Lemma 16, we can check the assertions by direct calculations. □

Now we proceed with the proof of Lemma 2. Let $S = V(F)$ be a surface containing C. Then F can be expressed as $\lambda_1 F_1 + \lambda_2 F_2$, where $(\lambda_1 : \lambda_2) \in \mathbb{P}^1$. In case $\lambda_2 = 0$, the point $Q_0 = (0 : 0 : 0 : 1)$ is the singular point of $V(F_1)$. On the other hand, in case $\lambda_2 \ne 0$, put $b = \lambda_1 / \lambda_2$. So we assume $F = bF_1 + F_2$. Consider the condition that $V(F)$ has a singular point, i.e., consider the simultaneous linear equations

$$F_X = F_Y = F_Z = F_W = 0. \tag{3}$$

This is equivalent to consider the rank of the matrix

$$M_b = \begin{pmatrix} 2q & b & p & 0 \\ b & 0 & 4 & 0 \\ p & 4 & -2b & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}. \tag{4}$$

The equations (3) have a non-trivial solution if and only if

$$b^3 + 4pb - 16q = 0. \tag{5}$$

It easy to see that the left hand side of (5) can be factored into $(b+4e_1)(b+4e_2)(b+4e_3)$. Thus, there exist three distinct solutions of (3). Since the rank of M_b is three for each solution of (3), each surface $S_i = V(b_iF_1 + F_2)$ is irreducible, where $b_i = -4e_i$. Let Q_i be the unique singular point of S_i. By simple calculations we obtain $Q_i = (8 : -2p - {b_i}^2 : -2b_i : 0) = (1 : -c_i : e_i : 0)$, where $c_i = {e_i}^2 + e_je_k$ such that $\{i, j, k\} = \{1, 2, 3\}$. Since

$$\det \begin{pmatrix} 1 & -c_1 & e_1 \\ 1 & -c_2 & e_2 \\ 1 & -c_3 & e_3 \end{pmatrix} = 2(e_1 - e_2)(e_2 - e_3)(e_3 - e_1) \neq 0,$$

the four points are not coplanar. This completes the proof.

The proof of Remark 3 is as follows. Let Σ_Q be the set $\{\ \ell_{QR} \mid R \in C\ \}$. Then there exists a cone S_Q with the singularity at Q such that $S_Q \supset C$ and $S_Q \supset \Sigma$. Therefore, by Lemma 2, we have $Q = Q_i$ for some i.

Combining Lemmas 1, 2 and 15, we infer readily Theorem 1.

Remark 18. By using the condition ($\diamondsuit$) in Lemma 12, we can prove that the number of V_4-lines is just six. However, Lemmas 1 and 2 give the more detailed structure of the arrangement of V_4-lines.

Now we go to the proof of Theorem 2. Since $j(C) = 1$, we can assume $\omega = \sqrt{-1}$. Hereafter, for simplicity we use i instead of $\sqrt{-1}$, so $\mathcal{L} = \mathbb{Z} + \mathbb{Z}i$.

Lemma 19. *The number of Z_4-lines is at most eight.*

Proof. Suppose C has a Z_4-line ℓ. Then, let H be a plane containing ℓ and $P_0{}'$. Since $\pi_\ell|_C : C \longrightarrow \mathbb{P}^1$ is a Z_4-covering, one of the following three cases take place:

(i) $H \cdot C = 4P_0{}'$.
(ii) $H \cdot C = 2P_0{}' + 2P_\gamma{}'$
(iii) $H \cdot C = P_0{}' + P_{\gamma_1}{}' + P_{\gamma_2}{}' + P_{\gamma_3}{}'$.

Suppose $G = \langle \sigma \rangle$, where

$$\sigma(z) = iz + \alpha \tag{6}$$

on the universal covering $\mathbb{C}$. The case (i) occurs if and only if $P_0{}'$ is a fixed point for σ, i.e., $\alpha \equiv 0 \pmod{\mathcal{L}}$ in (6). The case (ii) occurs if and only if $P_0{}'$ is a fixed point for σ^2, i.e., $2\alpha \equiv 0 \pmod{\mathcal{L}}$ in (6). Concerning the last case (iii), since $H \cdot C$ is invariant by the action of G, it can be expressed as $P_0{}' + P_\alpha{}' + P_{i\alpha}{}' + P_{(1+i)\alpha}{}'$. Since this is linearly equivalent to $4P_0{}'$, we infer

$$P_\alpha + P_{i\alpha} + P_{(1+i)\alpha} \sim 3P_0 \tag{7}$$

on the curve E. Moreover the left hand side of (7) is linearly equivalent to $P_{2(1+i)\alpha} + 2P_0$ by Lemma 8. Therefore we have $P_{2(1+i)}\alpha \sim P_0$. This implies $2(1+i)\alpha \equiv 0 \pmod{\mathcal{L}}$. To find the possibility of α, it is sufficient to solve the equation $2(1+i)\alpha \equiv 0 \pmod{\mathcal{L}}$. By a simple calculation we have $\alpha = (m + ni)/4$, where

$$(m, n) = (0,\ 0),\ (2,\ 2),\ (2,\ 0),\ (0,\ 2),\ (3,\ 1),\ (1,\ 3),\ (1,\ 1),\ (3,\ 3).$$

Thus we get eight subgroups, which might be the images of ρ of Definition 3. $\square$

Checking the condition ($\diamondsuit$) of Lemma 12, we now prove Theorem 2. As we see from the proof of Lemma 19, we have $G = \langle\sigma\rangle$, where $\sigma(z) = iz+\alpha$. Since σ has fixed points, the curve C/G is rational. For each point $Q \in C$ there exists $\gamma \in \mathbb{C}$ satisfying that $Q = P'_\gamma$. So it is sufficient to prove that ${P_\gamma}' + {P_{\sigma(\gamma)}}' + {P_{\sigma^2(\gamma)}}' + {P_{\sigma^3(\gamma)}}' \sim 4{P_0}'$. Since $2(1+i)\alpha \equiv 0 \pmod{\mathcal{L}}$ as in the proof of Lemma 19, this holds true by Lemma 8. Since $j(C) = 1$, we can assume $y^2 = 4x^3 - x$ and hence $e_1 = 1/2,\ e_2 = -1/2,\ e_3 = 0$. Thus we have $Q_0 = (0:0:0:1),\ Q_1 = (4:-1:2:0),\ Q_2 = (4:-1:-2:0)$ and $Q_3 = (4:1:0:0)$. Let ℓ_1 and ℓ_2 are Z_4-lines and $G_{\ell_1} = \langle\tau_1\rangle$ and $G_{\ell_2} = \langle\tau_2\rangle$. If ℓ_1 and ℓ_2 meet, then we have ${\tau_1}^2 = {\tau_2}^2$ by Lemma 13. Letting $\tau_1(z) = iz + \alpha_1$ and $\tau_2(z) = iz + \alpha_2$, we have $(1+i)(\alpha_1 - \alpha_2) \in \mathcal{L}$. Denote by $\ell(m,n)$ the line corresponding to the group $\langle\tau\rangle$ by the arrangement-map ρ, where $\tau(z) = iz + (m+ni)/4$. The following assertion is easy to see.

Claim 1. *Putting $\sigma_{mn}(z) = iz+(m+ni)/4$ and $G_{mn} = \langle\sigma_{mn}\rangle$, we have $G_{00}\cap G_{22} = \langle\sigma_0\rangle,\ G_{20}\cap G_{02} = \langle\sigma_3\rangle,\ G_{11}\cap G_{33} = \langle\sigma_2\rangle$ and $G_{31}\cap G_{13} = \langle\sigma_1\rangle$.*

Claim 2. *The intersections of the eight Z_4-lines are $\ell(0,0)\cap\ell(2,2) = Q_0$, $\ell(2,0)\cap\ell(0,2) = Q_3$, $\ell(1,1)\cap\ell(3,3) = Q_2$ and $\ell(3,1)\cap\ell(1,3) = Q_1$.*

Proof. The intersection points are found by Lemma 17. For example, the point $\ell(1,1)\cap\ell(3,3)$ is found as follows: Since $G_{11}\cap G_{33} = \langle\sigma_2\rangle$, the point is the intersection of two lines

$$c_3X - Y + 2e_1Z = W = 0 \text{ and } c_1X - Y + 2e_1 = W = 0,$$

where $e_1 = 1/2,\ e_3 = 0$ and $c_1 = 1/4,\ c_3 = -1/4$. So it is Q_2.

$\square$

Now, we prove Corollary 5. Let E be the Weierstrass normal form of the normalization of Γ and let $\mu : E \longrightarrow \Gamma \subset \mathbb{P}^3$ be the normalization morphism. Put $D = \mu^*(L)$ for a line L in $\mathbb{P}^2$. By Lemma 10 we can assume C is given by the embedding by $|4P_0|$. Therefore, Γ is regained as $\pi_P(C)$, where $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is the projection with center P. Suppose Γ has two Galois points Q_1 and Q_2. Then, letting $\ell_1 = {\pi_P}^*(Q_1)$ and $\ell_2 = {\pi_P}^*(Q_2)$, they are Galois lines for C and $\ell_1\cap\ell_2 = \{P\}$. However, as we have seen Remark 14, the projection π_P induces a 2 to 1 morphism from C to Γ and $\pi_P(C)$ is a rational curve, this is a contradiction. On the other hand, if P lies in one of the Galois lines, i.e., $P \in \ell$ and is not the vertex, then π_P induces a birational transformation on C by Remark 3 and $\pi_P(\ell\setminus\{P\})$ is a Galois point for $\Gamma = \pi_P(C)$.

Finally, we mention Remark 6. Take a point $Q \in \mathbb{P}^3$ which does not lie on the Galois lines. Then, the curve $\Gamma = \pi_Q(C)$ is a quartic curve with no Galois point. Because, by Remark 3 it is birational to C. Suppose it has a Galois point. Then, there exists a smooth quartic curve C' in $\mathbb{P}^3$ and a Galois line ℓ' and a point $P' \in \mathbb{P}^3$ satisfying that $\pi_{P'}(C') = \Gamma$. Moreover, there exists an isomorphism $\varphi : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ such that $\varphi(C') = C$ and $\varphi(\ell')$ coincides with some Galois line for C. Since $\ell' \ni P'$, we have $\varphi(\ell') \ni P$, which is a contradiction.

Thus we complete all proofs.

Problem. We ask the following questions concerning Galois embedding of elliptic curves.

(a) In case ℓ is not a Galois line, consider the Galois group G of the Galois closure curve [5, Definition 1.3]. If ℓ is general, then the Galois group is a full symmetric group [5, Theorem 2.2], see also [4]. So we ask if ℓ is neither general (i.e., $G \ncong S_4$) nor Galois, then what group can appear. For the group which appears, how are the arrangements of the lines with the group?

(b) Let D be a divisor of degree $d \geq 5$ on E. Then, study the Galois embedding by $|D|$. In particular, consider the Galois group and the arrangement of Galois subspaces ([6]).

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